

# Smoluchowski-Kramers approximation and large deviations for infinite dimensional gradient systems

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## Abstract

In this paper, we explicitly calculate the quasi-potentials for the damped semilinear stochastic wave equation when the system is of gradient type. We show that in this case the infimum of the quasi-potential with respect to all possible velocities does not depend on the density of the mass and does coincide with the quasi-potential of the corresponding stochastic heat equation that one obtains from the zero mass limit. This shows in particular that the Smoluchowski-Kramers approximation can be used to approximate long time behavior in the zero noise limit, such as exit time and exit place from a basin of attraction.

## 1 Introduction

In the present paper, we consider the following damped wave equation in a bounded regular domain  $\mathcal{O} \subset \mathbb{R}^d$ , perturbed by noise

$$\begin{cases} \mu \frac{\partial^2 u_\epsilon^\mu}{\partial t^2}(t, \xi) = \Delta u_\epsilon^\mu(t, \xi) - \frac{\partial u_\epsilon^\mu}{\partial t}(t, \xi) + B(u_\epsilon^\mu(t, \cdot))(\xi) + \sqrt{\epsilon} \frac{\partial w^Q}{\partial t}(t, \xi), & \xi \in \mathcal{O}, \ t > 0, \\ u_\epsilon^\mu(0, \xi) = u_0(\xi), \quad \frac{\partial u_\epsilon^\mu}{\partial t}(0, \xi) = v_0(\xi), \quad \xi \in \mathcal{O}, \quad u_\epsilon^\mu(t, \xi) = 0, \quad \xi \in \partial\mathcal{O}, \end{cases} \quad (1.1)$$

for some parameters  $0 < \epsilon, \mu \ll 1$ . Here,  $\partial w^Q / \partial t$  is a cylindrical Wiener process, white in time and colored in space, with covariance operator  $Q^2$ , for some  $Q \in \mathcal{L}(L^2(\mathcal{O}))$ . Concerning the non-linearity  $B$ , we assume

$$B(x) = -Q^2 DF(x), \quad x \in L^2(\mathcal{O}),$$

for some  $F : L^2(\mathcal{O}) \rightarrow \mathbb{R}$ , satisfying suitable conditions. We also consider the semi-linear heat equation

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t}(t, \xi) = \Delta u_\epsilon(t, \xi) + B(u_\epsilon(t, \cdot))(\xi) + \sqrt{\epsilon} \frac{\partial w^Q}{\partial t}(t, \xi), & \xi \in \mathcal{O}, \ t > 0, \\ u_\epsilon(0, \xi) = u_0(\xi), \quad u_\epsilon(t, \xi) = 0, \xi \in \partial\mathcal{O}. \end{cases} \quad (1.2)$$

In [1], it has been proved that if  $\epsilon > 0$  is fixed and  $\mu$  tends to zero, the solutions of (1.1) converge to the solution of (1.2), uniformly on compact intervals. More precisely, for any  $\eta > 0$  and  $T > 0$

$$\lim_{\mu \rightarrow 0} \mathbb{P} \left( \sup_{t \in [0, T]} |u_\epsilon^\mu(t) - u_\epsilon(t)|_{L^2(\mathcal{O})} > \eta \right) = 0.$$

Moreover, in the case  $d = 1$  and  $Q = I$ , it has been proven that for any fixed  $\epsilon > 0$ , the first marginal of the invariant measure of equation (1.1) coincides with the invariant measure of equation (1.2), for any  $\mu > 0$ .

In this paper, we are interested in comparing the small noise behavior of the two systems. More precisely, we keep  $\mu > 0$  fixed and let  $\epsilon$  tend to zero, to study some relevant quantities associated with the large deviation principle for these systems, as the quasi-potential that describes also the asymptotic behavior of the expected exit time from a domain and the corresponding exit places. Due to the gradient structure of (1.1), as in the finite dimensional case studied in [7], we are here able to calculate explicitly the quasi-potentials  $V^\mu(x, y)$  for system (1.1) as

$$V^\mu(x, y) = \left| (-\Delta)^{\frac{1}{2}} Q^{-1} x \right|_{L^2(\mathcal{O})}^2 + 2F(x) + \mu \left| Q^{-1} y \right|_{L^2(\mathcal{O})}^2. \quad (1.3)$$

Actually, we can prove that for any  $\mu > 0$

$$V^\mu(x, y) = \inf \left\{ I_{-\infty}^\mu(z) : z(0) = (x, y), \lim_{t \rightarrow -\infty} \left| C_\mu^{-1/2} z(t) \right|_H = 0 \right\}, \quad (1.4)$$

where

$$I_{-\infty}^\mu(z) = \frac{1}{2} \int_{-\infty}^0 \left| Q^{-1} \left( \mu \frac{\partial^2 \varphi}{\partial t^2}(t) - \Delta \varphi(t) + \frac{\partial \varphi}{\partial t}(t) - B(\varphi(t)) \right) \right|_{L^2(\mathcal{O})}^2 dt$$

with  $\varphi(t) = \Pi_1 z(t)$ , and

$$C_\mu(u, v) = \frac{1}{2} \left( (-\Delta)^{-1} Q^2 u, \frac{1}{\mu} (-\Delta)^{-1} Q^2 v \right), \quad (u, v) \in L^2(\mathcal{O}) \times H^{-1}(\mathcal{O}).$$

From (1.4), we obtain that

$$V^\mu(x, y) \geq \left| (-\Delta)^{\frac{1}{2}} Q^{-1} x \right|_{L^2(\mathcal{O})}^2 + 2F(x) + \mu \left| Q^{-1} y \right|_{L^2(\mathcal{O})}^2.$$

Thus, we obtain the equality (1.3) by constructing a path which realizes the minimum. An immediate consequence of (1.3) is that for each  $\mu > 0$

$$V_\mu(x) = \inf_{y \in H^{-1}(\mathcal{O})} V^\mu(x, y) = V^\mu(x, 0) = V(x), \quad (1.5)$$

where  $V(x)$  is the quasi-potential associated with equation (1.2).

Now, consider an open bounded domain  $G \subset L^2(D)$ , which is invariant for  $u_0^\mu$  and is attracted to the asymptotically stable equilibrium 0, and for any  $x \in G$  let us define

$$\tau_\epsilon^\mu := \inf \{ t \geq 0 : u_\epsilon^\mu(t) \in \partial G \} \text{ and } \tau_\epsilon = \inf \{ t \geq 0 : u_\epsilon(t) \in \partial G \}.$$

In a forthcoming paper we plan to prove

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E} \tau_\epsilon^\mu = \inf_{y \in \partial G} V_\mu(y) \quad (1.6)$$

As a consequence of (1.5), this would imply that, in the gradient case for any fixed  $\mu > 0$  it holds

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E} \tau_\epsilon^\mu = \inf_{y \in \partial G} V(y) = \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E} \tau_\epsilon^\mu.$$

## 2 Preliminaries and assumptions

Let  $H = L^2(\mathcal{O})$  and let  $A$  be the realization of the Laplacian with Dirichlet boundary conditions in  $H$ . Let  $\{e_k\}_{k \in \mathbb{N}}$  be the complete orthonormal basis of eigenvectors of  $A$ , and let  $\{-\alpha_k\}_{k \in \mathbb{N}}$  be the corresponding sequence of eigenvalues, with  $0 < \alpha_1 \leq \alpha_k \leq \alpha_{k+1}$ , for any  $k \in \mathbb{N}$ .

The stochastic perturbation is given by a cylindrical Wiener process  $w^Q(t, \xi)$ , for  $t \geq 0$  and  $\xi \in \mathcal{O}$ , which is assumed to be white in time and colored in space, in the case of space dimension  $d > 1$ . Formally, it is defined as the infinite sum

$$w^Q(t, \xi) = \sum_{k=1}^{+\infty} Q e_k(\xi) \beta_k(t), \quad (2.1)$$

where  $\{e_k\}_{k \in \mathbb{N}}$  is the complete orthonormal basis in  $L^2(\mathcal{O})$  which diagonalizes  $A$  and  $\{\beta_k(t)\}_{k \in \mathbb{N}}$  is a sequence of mutually independent standard Brownian motions defined on the same complete stochastic basis  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ .

**Hypothesis 1.** *The linear operator  $Q$  is bounded in  $H$  and diagonal with respect to the basis  $\{e_k\}_{k \in \mathbb{N}}$  which diagonalizes  $A$ . Moreover, if  $\{\lambda_k\}_{k \in \mathbb{N}}$  is the corresponding sequence of eigenvalues, we have*

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k} < +\infty. \quad (2.2)$$

In particular, if  $d = 1$  we can take  $Q = I$ , but if  $d > 1$  the noise has to be colored in space. Concerning the non-linearity  $B$ , we assume that it has the following gradient structure.

**Hypothesis 2.** *There exists  $F : H \rightarrow \mathbb{R}$  of class  $C^1$ , with  $F(0) = 0$ ,  $F(x) \geq 0$  and  $\langle DF(x), x \rangle \geq 0$  for all  $x \in H$ , such that*

$$B(x) = -Q^2 DF(x), \quad x \in H.$$

Moreover,

$$|DF(x) - DF(y)|_H \leq \kappa |x - y|_H, \quad x, y \in H. \quad (2.3)$$

**Example 2.1.** 1. Assume  $d = 1$  and take  $Q = I$ . Let  $b : \mathbb{R} \rightarrow \mathbb{R}$  be a decreasing Lipschitz continuous function with  $b(0) = 0$ . Then the composition operator  $B(x)(\xi) = b(x(\xi))$ ,  $\xi \in \mathcal{O}$ , is of gradient type. Actually, if we set

$$F(x) = - \int_{\mathcal{O}} \int_0^{x(\xi)} b(\eta) d\eta d\xi, \quad x \in H,$$

we have

$$B(x) = -DF(x), \quad x \in H.$$

Moreover, it is clear that  $F(0) = 0$ ,  $F(x) \geq 0$  for all  $x \in H$ , and

$$\langle DF(x), x \rangle = - \int_{\mathcal{O}} b(x(\xi))x(\xi) d\xi \geq 0, \quad x \in H.$$

2. Assume now  $d \geq 1$ , so that  $Q$  is a general bounded operator in  $H$ , satisfying Hypothesis 1. Let  $b : \mathbb{R} \rightarrow \mathbb{R}$  be a function of class  $C^1$ , with Lipschitz-continuous first derivative, such that  $b(0) = 0$  and  $b(\eta) \geq 0$ , for all  $\eta \in \mathbb{R}$ . Moreover, the only local minimum of  $b$  occurs at 0. Let

$$F(x) = \int_{\mathcal{O}} b(x(\xi))d\xi, \quad x \in H.$$

It is immediate to check that  $F(0) = 0$  and  $F(x) \geq 0$ , for all  $x \in H$ . Furthermore, for any  $x \in H$

$$DF(x)(\xi) = b'(x(\xi)), \quad \xi \in \mathcal{O}.$$

Therefore, the nonlinearity

$$B(x) = -Q^2 b'(x(\cdot)), \quad x \in H,$$

satisfies Hypothesis 2.

For any  $\delta \in \mathbb{R}$ , we denote by  $H^\delta$  the completion of  $C_0^\infty(\mathcal{O})$  in the norm

$$|u|_{H^\delta}^2 = \sum_{k=1}^{\infty} \alpha_k^\delta \langle u, e_k \rangle_H.$$

This is a Hilbert space, with the scalar product

$$\langle u, v \rangle_{H^\delta} = \sum_{k=1}^{\infty} \alpha_k^\delta \langle u, e_k \rangle_H \langle v, e_k \rangle_H.$$

In what follows, we shall define  $\mathcal{H}_\delta = H^\delta \times H^{\delta-1}$  and we shall set  $\mathcal{H} = \mathcal{H}_0$ .

Next, for  $\mu > 0$ , we define  $A_\mu : D(A_\mu) \subset \mathcal{H}_\delta \rightarrow \mathcal{H}_\delta$  by

$$A_\mu(u, v) = \left( v, \frac{1}{\mu}Au - \frac{1}{\mu}v \right), \quad (u, v) \in D(A_\mu) = \mathcal{H}_{\delta+1}, \quad (2.4)$$

and we denote by  $S_\mu(t)$  the semigroup on  $\mathcal{H}$  generated by  $A_\mu$ .

In [1, Proposition 2.4], it has been proven that  $S_\mu(t)$  is a  $C_0$ -semigroup of negative type, namely, there exist  $M_\mu > 0$  and  $\omega_\mu > 0$  such that

$$\|S_\mu(t)\|_{L(\mathcal{H}_\delta)} \leq M_\mu e^{-\omega_\mu t}, \quad t \geq 0. \quad (2.5)$$

Moreover, for any  $\mu > 0$  we define the operator  $Q_\mu : H^{\delta-1} \rightarrow \mathcal{H}_\delta$  by setting

$$Q_\mu v = \frac{1}{\mu} (0, Qv), \quad v \in H^{\delta-1}.$$

Therefore, if we define

$$\hat{F}(u, v) = F(u), \quad \hat{Q}(u, v) = Qu, \quad (u, v) \in \mathcal{H},$$

equation (1.1) can be rewritten as the following abstract stochastic evolution equation in the space  $\mathcal{H}$

$$dz_\epsilon^\mu(t) = \left[ A_\mu z_\epsilon^\mu(t) - Q_\mu \hat{Q} D\hat{F}(z_\epsilon^\mu(t)) \right] dt + \sqrt{\epsilon} Q_\mu dw(t), \quad z_\epsilon^\mu(0) = (u_0, v_0). \quad (2.6)$$

Analogously, equation (1.2) can be rewritten as the following abstract stochastic evolution equation in  $H$

$$du_\epsilon(t) = \left[ Au_\epsilon(t) - Q^2 DF(u_\epsilon(t)) \right] dt + \sqrt{\epsilon} Q dw(t), \quad u_\epsilon(0) = u_0. \quad (2.7)$$

**Definition 2.2.** 1. A predictable process  $z_\epsilon^\mu \in L^2(\Omega, C([0, T]; \mathcal{H}))$  is a mild solution to (2.6) if

$$z_\epsilon^\mu(t) = S_\mu(t)(u_0, v_0) - \int_0^t S_\mu(t-s) Q_\mu \hat{Q} D\hat{F}(z_\epsilon^\mu(s)) ds + \sqrt{\epsilon} \int_0^t S_\mu(t-s) Q_\mu dw(s).$$

2. A predictable process  $u^\epsilon \in L^2(\Omega, C([0, T]; H))$  is a mild solution to (2.7) if

$$u^\epsilon(t) = e^{tA} u_0 - \int_0^t e^{(t-s)A} Q^2 DF(u^\epsilon(s)) ds + \sqrt{\epsilon} \int_0^t e^{(t-s)A} Q dw(s).$$

**Remark 2.3.** If we define

$$C_\mu = \int_0^{+\infty} S_\mu(t) Q_\mu Q_\mu^* S_\mu^*(t) dt,$$

as shown in [1, Proposition 5.1] we have

$$C_\mu(u, v) = \frac{1}{2} \left( (-A)^{-1} Q^2 u, \frac{1}{\mu} (-A)^{-1} Q^2 v \right), \quad (u, v) \in \mathcal{H}.$$

Therefore, we get

$$2A_\mu C_\mu D\hat{F}(u, v) = (0, -\frac{1}{\mu} Q^2 DF(u)) = -Q_\mu \hat{Q} D\hat{F}(u, v), \quad (u, v) \in \mathcal{H}.$$

This means that equation (2.6) can be rewritten as

$$dz_\epsilon^\mu(t) = \left[ A_\mu z_\epsilon^\mu(t) + 2A_\mu C_\mu D\hat{F}(z_\epsilon^\mu(t)) \right] dt + \sqrt{\epsilon} Q_\mu dw(t), \quad z_\epsilon^\mu(0) = (u_0, v_0).$$

In the same way, equation (2.7) can be rewritten as

$$du_\epsilon(t) = \left[ Au_\epsilon(t) + 2AC DF(u_\epsilon(t)) \right] dt + \sqrt{\epsilon} Q dw(t), \quad u_\epsilon(0) = u_0,$$

where

$$C = \int_0^{+\infty} e^{tA} Q Q^* e^{tA^*} dt = \frac{1}{2} (-A)^{-1} Q^2.$$

In particular, both (2.6) and (2.7) are gradient systems (for more details see [6])

As we are assuming that  $DF : H \rightarrow H$  is Lipschitz continuous, we have that equation (2.6) has a unique mild solution  $z_\epsilon^\mu \in L^p(\Omega, C([0, T]; \mathcal{H}))$  and equation (2.7) has a unique mild solution  $u^\epsilon \in L^p(\Omega, C([0, T]; H))$ , for any  $p \geq 1$  and  $T > 0$ .

In [1, Theorem 4.6] it has been proved that in this case the so-called Smoluchowski-Kramers approximation holds. Namely, for any  $\epsilon, T > 0$  and  $\eta > 0$

$$\lim_{\mu \rightarrow 0} \mathbb{P} \left( \sup_{t \in [0, T]} |u_\epsilon^\mu(u) - u_\epsilon(t)|_H > \eta \right) = 0,$$

where  $u_\epsilon^\mu(t) = \Pi_1 z_\epsilon^\mu(t)$ .

### 3 A characterization of the quasi-potential

For any  $\mu > 0$  and  $t_1 < t_2$ , and for any  $z \in C([t_1, t_2]; \mathcal{H})$  and  $z_0 \in \mathcal{H}$ , we define

$$I_{t_1, t_2}^\mu(z) = \frac{1}{2} \inf \left\{ |\psi|_{L^2((t_1, t_2); H)}^2 : z = z_{z_0, \psi}^\mu \right\} \quad (3.1)$$

where  $z_{z_0, \psi}^\mu$  solves the following skeleton equation associated with (2.6)

$$z_{z_0, \psi}^\mu(t) = S_\mu(t - t_1)z_0 + \int_{t_1}^t S_\mu(t - s)A_\mu C_\mu D\hat{F}_\mu(z_{z_0, \psi}^\mu(s))ds + \int_{t_1}^t S_\mu(t - s)Q_\mu \psi(s)ds. \quad (3.2)$$

Analogously, for  $t_1 < t_2$ , and for any  $\varphi \in C([t_1, t_2]; H)$  and  $u_0 \in H$ , we define

$$I_{t_1, t_2}(\varphi) = \frac{1}{2} \inf \left\{ |\psi|_{L^2((t_1, t_2); H)}^2 : \varphi = \varphi_{\psi, u_0} \right\}, \quad (3.3)$$

where  $\varphi_{u_0, \psi}$  solves the problem

$$\varphi_{u_0, \psi}(t) = e^{(t-t_1)A}u_0 - \int_{t_1}^t e^{(t-s)A}Q^2 DF(\varphi_{u_0, \psi}(s))ds + \int_{t_1}^t e^{(t-s)A}Q\psi(s)ds. \quad (3.4)$$

In what follows, we shall also denote

$$I_{-\infty}^\mu(z) = \sup_{t < 0} I_{t, 0}^\mu(z), \quad I_{-\infty}(\varphi) = \sup_{t < 0} I_{t, 0}(\varphi).$$

Since (2.6) and (2.7) have additive noise, as a consequence of the contraction lemma we have that the family  $\{z_\epsilon^\mu\}_{\epsilon > 0}$  satisfies the large deviations principle in  $C([0, T]; \mathcal{H})$ , with respect to the rate function  $I_{0, T}^\mu$  and the family  $\{u^\epsilon\}_{\epsilon > 0}$  satisfies the large deviations principle in  $C([0, T]; H)$ , with respect to the rate function  $I_{0, T}$ .

In what follows, for any fixed  $\mu > 0$  we shall denote by  $V^\mu$  the quasi-potential associated with system (2.6), namely

$$V^\mu(x, y) = \inf \left\{ I_{0, T}^\mu(z) : z(0) = 0, z(T) = (x, y), T > 0 \right\}.$$

Analogously, we shall denote by  $V$  the quasi-potential associated with equation (2.7), that is

$$V(x) = \inf \{ I_{0, T}(\varphi) : \varphi(0) = 0, \varphi(T) = x, T > 0 \}.$$

Moreover, for any  $\mu > 0$  we shall define

$$V_\mu(x) = \inf_{y \in H^{-1}(\mathcal{O})} V^\mu(x, y) = \inf \left\{ I_{0,T}^\mu(z) : z(0) = 0, \Pi_1 z(T) = x, T > 0 \right\}.$$

In [3, Proposition 5.4], it has been proven that  $V(x)$  can be represented as

$$V(x) = \inf \left\{ I_{-\infty}(\varphi) : \varphi(0) = x, \lim_{t \rightarrow -\infty} |\varphi(t)|_H = 0 \right\}.$$

Here, we want to prove a similar representations for  $V^\mu(x, y)$ , for any fixed  $\mu > 0$ . To this purpose, we first introduce the operator  $L_{t_1, t_2}^\mu : L^2((t_1, t_2); H) \rightarrow \mathcal{H}$ , defined as

$$L_{t_1, t_2}^\mu \psi = \int_{t_1}^{t_2} S_\mu(t_2 - s) Q_\mu \psi(s) ds. \quad (3.5)$$

**Theorem 3.1.** *For any  $\mu > 0$  and  $(x, y) \in \mathcal{H}$  we have*

$$V^\mu(x, y) = \inf \left\{ I_{-\infty}^\mu(z) : z(0) = (x, y), \lim_{t \rightarrow -\infty} \left| C_\mu^{-1/2} z(t) \right|_H = 0 \right\}. \quad (3.6)$$

*Proof.* First we observe that by the definitions of  $I_{t_1, t_2}^\mu$  and  $V^\mu(x, y)$ ,

$$V^\mu(x, y) = \inf \left\{ I_{-T, 0}^\mu(z) : z(-T) = 0, z(0) = (x, y), T > 0 \right\}.$$

Now, for any  $\mu > 0$  and  $(x, y) \in \mathcal{H}$ , we define

$$M^\mu(x, y) := \inf \left\{ I_{-\infty}^\mu(z) : z(0) = (x, y), \lim_{t \rightarrow -\infty} \left| C_\mu^{-1/2} z(t) \right|_{\mathcal{H}} = 0 \right\}.$$

Clearly, we want to prove that  $M^\mu(x, y) = V^\mu(x, y)$ , for all  $(x, y) \in \mathcal{H}$ .

If  $z$  is a continuous path with  $z(-T) = 0$  and  $z(0) = (x, y)$ , we can extend it in  $C((-\infty, 0); \mathcal{H})$ , by defining  $z(t) = 0$ , for  $t < -T$ . Then, since  $DF(0) = 0$ , we see that

$$I_{-\infty}^\mu(z) = I_{-T, 0}^\mu(z),$$

so that  $M^\mu(x, y) \leq V^\mu(x, y)$ .

Now, let us prove that the opposite inequality holds.  $M^\mu(x, y) = +\infty$ , there is nothing else to prove. Thus, we assume that  $M^\mu(x, y) < +\infty$ . This means that for any  $\epsilon > 0$  there must be some  $z_\epsilon^\mu \in C((-\infty, 0); \mathcal{H})$ , with the properties that  $z_\epsilon^\mu(0) = (x, y)$  and

$$\lim_{t \rightarrow -\infty} |C_\mu^{-1/2} z_\epsilon^\mu(t)|_{\mathcal{H}} = 0, \quad I_{-\infty}^\mu(z_\epsilon^\mu) \leq M^\mu(x, y) + \epsilon.$$

In what follows we shall prove that the following auxiliary result holds.

**Lemma 3.2.** *For any  $\mu > 0$ , there exists  $T_\mu > 0$  such that for any  $t_1 < t_2 - T_\mu$  we have  $\text{Im}(L_{t_1, t_2}^\mu) = D(C_\mu^{-1/2})$  and*

$$\left| (L_{t_1, t_2}^\mu)^{-1} z \right|_{L^2((t_1, t_2); H)} \leq c(\mu, t_2 - t_1) \left| C_\mu^{-1/2} z \right|_{\mathcal{H}}, \quad z \in \text{Im}(L_{t_1, t_2}^\mu), \quad (3.7)$$

where

$$C_\mu(x, y) = \left( (-A)^{-1} Q^2 x, \frac{1}{\mu} (-A)^{-1} Q^2 y \right).$$

Thus, let  $T_\mu$  be the constant from Lemma 3.2 and let  $t_\epsilon < 0$  be such that

$$|C_\mu^{-1/2} z_\epsilon^\mu(t_\epsilon)|_{\mathcal{H}} < \epsilon.$$

Moreover, let  $\psi_\epsilon^\mu := (L_{t_\epsilon - T_\mu, t_\epsilon}^\mu)^{-1} z_\epsilon^\mu(t_\epsilon)$ . By Lemma 3.2 we have

$$|\psi_\epsilon^\mu|_{L^2(t_\epsilon - T_\mu, t_\epsilon; H)} \leq c_\mu \epsilon.$$

Next, we define

$$\hat{z}_\epsilon^\mu(t) = \int_{t_\epsilon - T_\mu}^t S_\mu(t-s) Q_\mu \psi_\epsilon^\mu(s) ds.$$

Thanks to (2.5), we have

$$\begin{aligned} \int_{t_\epsilon - T_\mu}^{t_\epsilon} |\hat{z}_\epsilon^\mu(t)|_{\mathcal{H}}^2 dt &\leq \int_{t_\epsilon - T_\mu}^{t_\epsilon} \left( \int_{t_\epsilon - T_\mu}^t \frac{M_\mu}{\mu} e^{-\omega_\mu(t-s)} |Q \psi_\epsilon^\mu(s)|_{H^{-1}} ds \right)^2 dt \\ &\leq \frac{M_\mu^2}{2\mu^2 \omega_\mu} \int_{t_\epsilon - T_\mu}^{t_\epsilon} |Q \psi_\epsilon^\mu|_{L^2((t_\epsilon - T_\mu, t_\epsilon); H^{-1})}^2 dt \leq T_\mu \frac{M_\mu^2}{2\mu^2 \omega_\mu} |Q \psi_\epsilon^\mu|_{L^2((t_\epsilon - T_\mu, t_\epsilon); H^{-1})}^2 \leq c_\mu \epsilon^2. \end{aligned}$$

Furthermore,  $\hat{z}_\epsilon^\mu(t_\epsilon - T_\mu) = 0$  and  $\hat{z}_\epsilon^\mu(t_\epsilon) = z(t_\epsilon)$ . Finally, we notice that

$$\hat{z}_\epsilon^\mu(t) = - \int_{t_\epsilon - T_\mu}^t S_\mu(t-s) Q_\mu Q DF(\hat{z}_\epsilon^\mu(s)) ds + \int_{t_\epsilon - T_\mu}^t S_\mu(t-s) Q_\mu (\psi_\epsilon^\mu(s) + Q DF(\hat{z}_\epsilon^\mu(s))) ds,$$

so that

$$I_{t_\epsilon - T_\mu, t_\epsilon}^\mu(\hat{z}_\epsilon^\mu) = \frac{1}{2} |\psi_\epsilon^\mu + Q DF(\hat{z}_\epsilon^\mu(s))|_{L^2((t_\epsilon - T_\mu, t_\epsilon); H)}^2 \leq c_\mu \epsilon^2.$$

Now if we define

$$\tilde{z}_\epsilon^\mu(t) = \begin{cases} \hat{z}_\epsilon^\mu(t) & \text{if } t_\epsilon - T_\mu \leq t < t_\epsilon \\ z_\epsilon^\mu(t) & \text{if } t_\epsilon \leq t \leq 0, \end{cases}$$

we see that  $\tilde{z}_\epsilon^\mu \in C((t_\epsilon - T_\mu, 0); \mathcal{H})$  and

$$V^\mu(x, y) \leq I_{t_\epsilon - T_\mu, 0}^\mu(\tilde{z}_\epsilon^\mu) = I_{t_\epsilon - T_\mu, t_\epsilon}^\mu(\hat{z}_\epsilon^\mu) + I_{t_\epsilon, 0}^\mu(z_\epsilon^\mu) \leq c_\mu \epsilon^2 + M^\mu(x, y) + \epsilon.$$

Due to the arbitrariness of  $\epsilon > 0$ , we can conclude.  $\square$

*Proof of Lemma 3.2.* It is immediate to check that

$$|(L_{t_1, t_2}^\mu)^\star z|_{L^2((t_1, t_2); H)}^2 = \frac{1}{\mu} \int_0^{t_2 - t_1} |Q_\mu^\star S_\mu^\star(s) z|_H^2 ds.$$

therefore, since

$$Q_\mu^\star(u, v) = \frac{1}{\mu} (-A)^{-1} Q v, \quad (u, v) \in \mathcal{H},$$

(see [1, Section 5] for a proof), we can conclude

$$|(L_{t_1, t_2}^\mu)^\star z|_{L^2((t_1, t_2); H)}^2 = \frac{1}{\mu^2} \int_0^{t_2 - t_1} |Q(-A)^{-1} \Pi_2 S_\mu^\star(s) z|_H^2 ds.$$



Now, if we expand  $S_\mu^\star(t)$  into Fourier coefficients, by [1, Proposition 2.3], we get

$$S_\mu^\star(u, v) = \sum_{k=1}^{\infty} \left( \hat{f}_k^\mu(t) e_k, \hat{g}_k^\mu(t) e_k \right)$$

where

$$\begin{cases} \mu \frac{d\hat{f}_k^\mu}{dt}(t) = -\hat{g}_k^\mu(t), & \hat{f}_k^\mu(0) = u_k = \langle u, e_k \rangle_H \\ \mu \frac{d\hat{g}_k^\mu}{dt}(t) = \mu \alpha_k \hat{f}_k^\mu(t) - \hat{g}_k^\mu(t), & \hat{g}_k^\mu(0) = v_k = \langle v, e_k \rangle_H. \end{cases}$$

From these equations we see that

$$|\hat{g}_k^\mu(t)|^2 = -\frac{\mu^2 \alpha_k}{2} \frac{d}{dt} |\hat{f}_k^\mu(t)|^2 - \frac{\mu}{2} \frac{d}{dt} |\hat{g}_k^\mu(t)|^2.$$

This means that

$$\begin{aligned} & |(L_{t_1, t_2}^\mu)^\star z|_{L^2((t_1, t_2); H)}^2 \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{\lambda_k^2}{\alpha_k} |u_k|^2 + \frac{\lambda_k^2}{\mu \alpha_k^2} |v_k|^2 - \frac{\lambda_k^2}{\alpha_k} |\hat{f}_k^\mu(t_2 - t_1)|^2 - \frac{\lambda_k^2}{\mu \alpha_k^2} |\hat{g}_k^\mu(t_2 - t_1)|^2 \right) \\ &= \frac{1}{2} \left( \left| C_\mu^{1/2} z \right|_{\mathcal{H}}^2 - \left| C_\mu^{1/2} S_\mu^\star(t_2 - t_1) z \right|_{\mathcal{H}}^2 \right). \end{aligned} \tag{3.8}$$

Notice that

$$(1 \wedge \sqrt{\mu}) \left| C_\mu^{1/2} z \right|_{\mathcal{H}} \leq \left| C_1^{1/2} z \right|_{\mathcal{H}} \leq (1 + \sqrt{\mu}) \left| C_\mu^{1/2} z \right|_{\mathcal{H}}$$

and that  $C_1^{1/2}$  commutes with  $S_\mu^\star(t)$ . Therefore, by (2.5)

$$\left| C_\mu^{1/2} S_\mu^\star(t) z \right|_{\mathcal{H}} \leq \frac{1}{1 \wedge \sqrt{\mu}} \left| S_\mu^\star(t) C_1^{1/2} z \right|_{\mathcal{H}} \leq \frac{1 + \sqrt{\mu}}{1 \wedge \sqrt{\mu}} M_\mu e^{-\omega_\mu t} \left| C_\mu^{1/2} z \right|_{\mathcal{H}}.$$

According to (3.8), this implies

$$|(L_{t_1, t_2}^\mu)^\star z|_{L^2((t_1, t_2); H)}^2 \geq \frac{1}{2} \left( 1 - \left( \frac{1 + \sqrt{\mu}}{1 \wedge \sqrt{\mu}} \right)^2 M_\mu^2 e^{-2\omega_\mu(t_2 - t_1)} \right) \left| C_\mu^{1/2} z \right|_{\mathcal{H}}^2,$$

so that, if we take  $T_\mu > 0$  such that

$$\left( \frac{1 + \sqrt{\mu}}{1 \wedge \sqrt{\mu}} \right)^2 M_\mu^2 e^{-2\omega_\mu T_\mu} < 1$$

we can conclude. □

## 4 The main result

If  $z \in C((-\infty, 0]; \mathcal{H})$  is such that  $I_{-\infty, 0}^\mu(z) < +\infty$ , then we have

$$I_{-\infty}^\mu(z) = \frac{1}{2} \int_{-\infty}^0 \left| Q^{-1} \left( \mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t)) \right) \right|_H^2 dt. \quad (4.1)$$

Actually, if  $I_{-\infty, 0}^\mu(z) < +\infty$ , then there exists  $\psi \in L^2((-\infty, 0); H)$  such that  $\varphi = \Pi_1 z$  is a weak solution to

$$\mu \frac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) - \frac{\partial \varphi}{\partial t}(t) - Q^2 DF(\varphi(t)) + Q\psi.$$

This means that

$$\psi(t) = Q^{-1} \left( \mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t)) \right)$$

and (4.1) follows.

By the same argument, if  $I_{-\infty, 0}(\varphi) < +\infty$ , then it follows that

$$I_{-\infty}(\varphi) = \int_{-\infty}^0 \left| Q^{-1} \left( \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t)) \right) \right|_H^2 dt \quad (4.2)$$

**Theorem 4.1.** *For any fixed  $\mu > 0$  and  $(x, y) \in D((-A)^{1/2}Q^{-1}) \times D(Q^{-1})$  it holds*

$$V^\mu(x, y) = \left| (-A)^{\frac{1}{2}} Q^{-1} x \right|_H^2 + 2F(x) + \mu \left| Q^{-1} y \right|_H^2. \quad (4.3)$$

Moreover

$$V(x) = \left| (-A)^{\frac{1}{2}} Q^{-1} x \right|_H^2 + 2F(x). \quad (4.4)$$

In particular, for any  $\mu > 0$ ,

$$V_\mu(x) := \inf_{y \in H^{-1}} V^\mu(x, y) = V^\mu(x, 0) = V(x).$$

*Proof.* First, we observe that if  $\varphi(t) = \Pi_1 z(t)$ , then

$$\begin{aligned} I_{-\infty}^\mu(z) &= \frac{1}{2} \int_{-\infty}^0 \left| Q^{-1} \left( \mu \frac{\partial \varphi}{\partial t}(t) - \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t)) \right) \right|_H^2 dt \\ &+ 2 \int_{-\infty}^0 \left\langle Q^{-1} \frac{\partial \varphi}{\partial t}(t), Q^{-1} \left( \mu \frac{\partial^2 \varphi}{\partial t^2}(t) - A\varphi(t) \right) + Q DF(\varphi(t)) \right\rangle_H dt. \end{aligned} \quad (4.5)$$

Now, if

$$\lim_{t \rightarrow -\infty} |C_\mu^{-1/2} z(t)|_{\mathcal{H}} = 0,$$

then

$$\lim_{t \rightarrow -\infty} \left| (-A)^{\frac{1}{2}} Q^{-1} \varphi(t) \right|_H + \left| Q^{-1} \frac{\partial \varphi}{\partial t}(t) \right|_H = 0,$$

so that

$$\begin{aligned} & 2 \int_{-\infty}^0 \left\langle Q^{-1} \frac{\partial \varphi}{\partial t}(t), Q^{-1} \left( \mu \frac{\partial^2 \varphi}{\partial t^2}(t) - A\varphi(t) \right) + QDF(\varphi(t)) \right\rangle_H dt \\ &= \left| (-A)^{\frac{1}{2}} Q^{-1} \varphi(0) \right|_H^2 + 2F(\varphi(0)) + \mu \left| Q^{-1} \frac{\partial \varphi}{\partial t}(0) \right|_H^2. \end{aligned}$$

This yields

$$V^\mu(x, y) \geq \left| (-A)^{\frac{1}{2}} Q^{-1} x \right|_H^2 + 2F(x) + \mu \left| Q^{-1} y \right|_H^2.$$

Now, let  $\tilde{z}(t)$  be a mild solution of the problem

$$\tilde{z}(t) = S_\mu(t)(x, -y) - \int_0^t S_\mu(t-s) Q_\mu QDF(\tilde{z}(s)) ds,$$

and let  $(x, y) \in D(C_\mu^{-1/2})$ . Then  $\tilde{\varphi}(t) = \Pi_1 \tilde{z}(t)$  is a weak solution of the problem

$$\mu \frac{\partial^2 \tilde{\varphi}}{\partial t^2}(t) = A\tilde{\varphi}(t) - \frac{\partial \tilde{\varphi}}{\partial t}(t) - Q^2 DF(\tilde{\varphi}(t)), \quad \tilde{\varphi}(0) = x, \quad \frac{\partial \tilde{\varphi}}{\partial t}(0) = -y.$$

Moreover, as proven below in Lemma 4.2,

$$\lim_{t \rightarrow -\infty} \left| C_\mu^{-1/2} \tilde{z}(t) \right|_{\mathcal{H}} = 0.$$

Then, if we define  $\hat{\varphi}(t) = \tilde{\varphi}(-t)$  for  $t \leq 0$ , we see that  $\hat{\varphi}(t)$  solves

$$\mu \frac{\partial^2 \hat{\varphi}}{\partial t^2}(t) = A\hat{\varphi}(t) + \frac{\partial \hat{\varphi}}{\partial t}(t) - Q^2 DF(\hat{\varphi}(t)), \quad \hat{\varphi}(0) = x, \quad \frac{\partial \hat{\varphi}}{\partial t}(0) = y.$$

Thanks to (4.5) this yields

$$I_{-\infty}^\mu(\hat{\varphi}) = \left| (-A)^{\frac{1}{2}} Q^{-1} x \right|_H^2 + 2F(x) + \mu \left| Q^{-1} y \right|_H^2.$$

and then

$$V^\mu(x, y) = \left| (-A)^{\frac{1}{2}} Q^{-1} x \right|_H^2 + 2F(x) + \mu \left| Q^{-1} y \right|_H^2.$$

As known, an analogous result holds for  $V(x)$ . In what follows, for completeness, we give a proof. We have

$$\begin{aligned} I_{-\infty}(\varphi) &= \frac{1}{2} \int_{-\infty}^0 \left| Q^{-1} \left( \frac{\partial \varphi}{\partial t}(t) + A\varphi(t) - Q^2 DF(\varphi(t)) \right) \right|_H^2 dt \\ &+ 2 \int_{-\infty}^0 \left\langle Q^{-1} \frac{\partial \varphi}{\partial t}(t), Q^{-1} (-A\varphi(t) + Q^2 DF(\varphi(t))) \right\rangle_H dt. \end{aligned} \tag{4.6}$$

From this we see that

$$V(x) \geq \left| (-A)^{\frac{1}{2}} Q^{-1} x \right|_H^2 + 2F(x).$$

Just as for the wave equation, for  $x \in D((-A)^{\frac{1}{2}}Q^{-1})$ , we define  $\tilde{\varphi}$  to be the solution of

$$\tilde{\varphi}(t) = e^{tA}x - \int_0^t e^{(t-s)A}Q^2DF(\tilde{\varphi}(s))ds.$$

We have

$$\lim_{t \rightarrow +\infty} |(-A)^{\frac{1}{2}}Q^{-1}\tilde{\varphi}(t)|_H = 0.$$

Then, if we define  $\hat{\varphi}(t) = \tilde{\varphi}(-t)$  we get

$$\frac{\partial \hat{\varphi}}{\partial t}(t) = -A\hat{\varphi}(t) + Q^2DF(\hat{\varphi}(t)),$$

so that

$$I_{-\infty}(\hat{\varphi}) = \left| (-A)^{\frac{1}{2}}Q^{-1}x \right|_H^2 + 2F(x)$$

and

$$V(x) = \left| (-A)^{\frac{1}{2}}Q^{-1}x \right|_H^2 + 2F(x).$$

□

Now, in order to conclude the proof of Theorem 4.1, we have to prove the following result.

**Lemma 4.2.** *Let  $(x, y) \in D((-A)^{\frac{1}{2}}Q^{-1}) \times D(Q^{-1})$  and let  $\varphi$  solve the problem*

$$\mu \frac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) - \frac{\partial \varphi}{\partial t}(t) - Q^2DF(\varphi(t)), \quad \varphi(0) = x, \quad \frac{\partial \varphi}{\partial t}(0) = y. \quad (4.7)$$

Then

$$z(t) = \left( \varphi(t), \frac{\partial \varphi}{\partial t}(t) \right) \in D((-A)^{\frac{1}{2}}) \times D(Q^{-1}), \quad t \geq 0,$$

and

$$\lim_{t \rightarrow +\infty} \left| C_1^{-1/2}z(t) \right|_{\mathcal{H}} = 0. \quad (4.8)$$

*Proof.* If in (4.7) we take the inner product with  $2Q^{-2}\partial\varphi/\partial t(t)$ , we have

$$2 \left| Q^{-1} \frac{\partial \varphi}{\partial t}(t) \right|_H^2 = -\frac{d}{dt} \left( \mu |Q^{-1}\varphi(t)|_H^2 + \left| (-A)^{\frac{1}{2}}Q^{-1}\varphi(t) \right|_H^2 + 2F(\varphi(t)) \right). \quad (4.9)$$

Therefore, if we define

$$\Phi_\mu(x, y) = \left| (-A)^{\frac{1}{2}}Q^{-1}x \right|_H^2 + \mu |Q^{-1}y|_H^2 + 2F(x),$$

as a consequence of (4.9) we get

$$\Phi_\mu(z(t)) \leq \Phi_\mu(u, v). \quad (4.10)$$

Next, by (4.7) and the assumption that  $\langle DF(x), x \rangle \geq 0$ , we calculate that

$$\begin{aligned} \frac{d}{dt} \left| Q^{-1} \left( \mu \frac{\partial \varphi}{\partial t}(t) + \varphi(t) \right) \right|_H^2 &= 2 \left\langle Q^{-1} \left( \mu \frac{\partial \varphi}{\partial t}(t) + \varphi(t) \right), Q^{-1} A \varphi(t) - Q DF(\varphi(t)) \right\rangle_H \\ &\leq -2 \left| Q^{-1}(-A)^{\frac{1}{2}} \varphi(t) \right|_H^2 - \mu \frac{d}{dt} \left| Q^{-1}(-A)^{\frac{1}{2}} \varphi(t) \right|_H^2 - 2\mu \frac{d}{dt} F(\varphi(t)). \end{aligned}$$

A consequence of this is that

$$2 \int_0^\infty \left| Q^{-1}(-A)^{\frac{1}{2}} \varphi(t) \right|_H^2 dt \leq |\mu Q^{-1}y + Q^{-1}x|_H^2 + \mu \left| Q^{-1}(-A)^{\frac{1}{2}}x \right|_H^2 + 2\mu F(x). \quad (4.11)$$

Now, if  $z(t) = \left( \varphi(t), \frac{\partial \varphi}{\partial t}(t) \right)$ , for any  $t, T > 0$  we have

$$z(T+t) = S_\mu(t)z(T) - \int_T^{T+t} S_\mu(T+t-s)Q_\mu Q DF(\varphi(s))ds.$$

By (2.5), and (2.3) we have

$$\begin{aligned} &\left| C_1^{-1/2} \int_T^{T+t} S_\mu(t+T-s)Q_\mu Q DF(\varphi(s))ds \right|_{\mathcal{H}} \\ &\leq \int_T^{T+t} \left| S_\mu(t+T-s)Q_\mu(-A)^{\frac{1}{2}} DF(\varphi(s)) \right|_{\mathcal{H}} ds \\ &\leq c \int_T^{T+t} e^{-\omega_\mu(t+T-s)} \left| (-A)^{\frac{1}{2}} Q DF(\varphi(s)) \right|_{H^{-1}} ds \\ &\leq c \int_T^{T+t} e^{-\omega_\mu(t+T-s)} |\varphi(s)|_H ds \leq c |\varphi|_{L^2((T, T+t); H)}. \end{aligned}$$

Therefore, by (4.11), for any  $\epsilon > 0$  we can pick  $T_\epsilon > 0$  large enough so that for all  $t > 0$

$$\left| C_1^{-1/2} \int_{T_\epsilon}^{T_\epsilon+t} S_\mu(t+T_\epsilon-s)Q_\mu Q DF(\varphi(s))ds \right|_{\mathcal{H}} < \frac{\epsilon}{2}.$$

Next, by (2.5),

$$\left| C_1^{-1/2} S_\mu(t)z(T) \right|_{\mathcal{H}} \leq M_\mu e^{-\omega_\mu t} \left| C_1^{-1/2} z(T) \right|_{\mathcal{H}}.$$

Then, as

$$|C_1^{-1/2} z|_{\mathcal{H}} \leq c \Phi_\mu(z), \quad z \in \mathcal{H},$$

by (4.10) we can find a  $t_\epsilon$  large enough so that for all  $T > 0$  and  $t > t_\epsilon$

$$\left| C_1^{-1/2} S_\mu(t)z(T) \right|_{\mathcal{H}} < \frac{\epsilon}{2}.$$

Then for  $t > T_\epsilon + t_\epsilon$

$$\left| C_1^{-1/2} z(t) \right|_{\mathcal{H}} < \epsilon$$

which is what we were trying to prove.  $\square$

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